ST2302/MA8002 Exercises chapter 2 and 3

Exercise 2.1

Assume the stochastic growth rate $r(t)$ has expectation r and variance σ_r^2. First, consider the expectation of ΔN, given by

$$\mathrm{E}[\Delta N] = r\,N\left(1 - \frac{N}{K}\right)$$

This is equivalent to the deterministic version of the logistic model, hence it seems reasonable. The variance of ΔN is given by

$$\mathrm{Var}(\Delta N) = N^2\left(1 - \frac{N}{K}\right)^2 \sigma_r^2$$

This variance depends on N. Figure 1 shows how it changes with N, for $K = 70$ and $\sigma_r^2 = 0.1$. It does not seem realistic that the variance should decline as $N \to K$.

Exercise 2.4

The theta-logistic model is given by

$$r(N) = \begin{cases} r_1\left(1 - \frac{N^\theta - 1}{K^\theta - 1}\right) & ,\ \theta \neq 0 \\ r_1\left(1 - \frac{\ln N}{\ln K}\right) & ,\ \theta = 0 \end{cases}$$

In order to verify the different cases, we will insert the different values of θ in this model.

1. First, consider the case where $\theta = -1$.

$$\begin{aligned} r(N) &= r_1\left(1 - \frac{\frac{1}{N} - 1}{\frac{1}{K} - 1}\right) \\ &= r_1\left(1 - \frac{\frac{K}{N} - K}{1 - K}\right) \\ &= r_1\left(\frac{1 - K - \frac{K}{N} + K}{1 - K}\right) \\ &= r_1\left(\frac{1 - \frac{K}{N}}{1 - K}\right) \end{aligned}$$

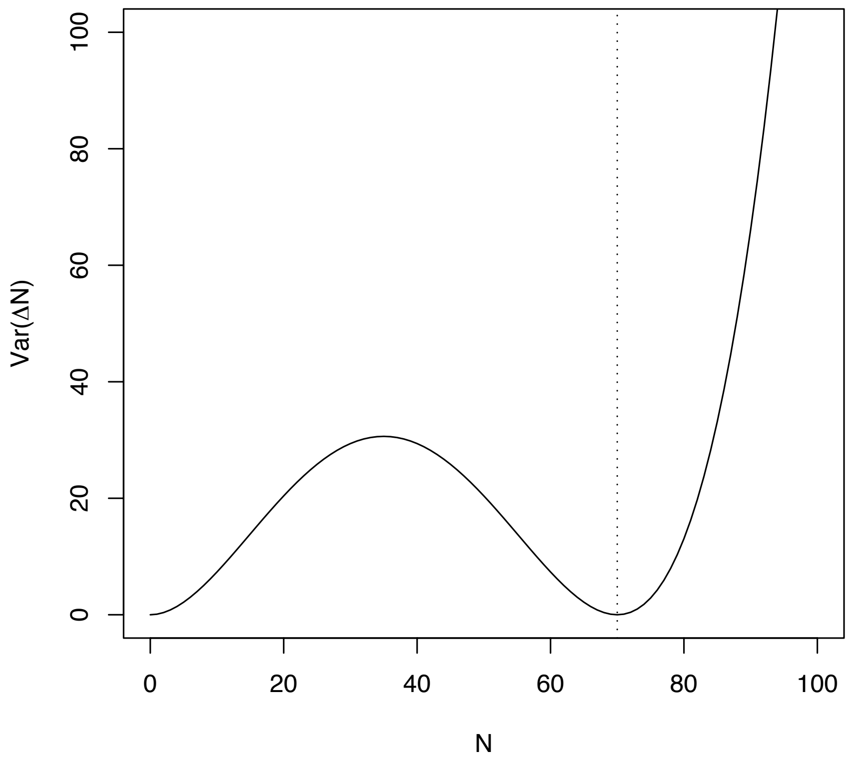

Figure 1: From exercise 2.1. How the variance in ΔN changes with N, for $K = 70$ (dashed line) and $\sigma_r^2 = 0.1$.

$$
\begin{aligned}
r(N) &= \Delta \ln N \approx \frac{\Delta N}{N} \\
\Delta N &\approx r_1\, N \left(\frac{1 - \frac{K}{N}}{1 - K} \right) \\
&= r_1 \left(\frac{N - K}{1 - K} \right)
\end{aligned}
$$

Hence, in this case ΔN is a linear function of N.

2. For the case of $\theta = 0$, we use l'Hôpital's rule to find the limit as $\theta \to 0$:

$$
r(N) = \lim_{\theta \to 0} r_1 \left(1 - \frac{N^\theta - 1}{K^\theta - 1} \right) = \lim_{\theta \to 0} r_1 \left(1 - \frac{N^\theta \ln N}{K^\theta \ln K} \right) = r_1 \left(1 - \frac{\ln N}{\ln K} \right)
$$

This is Gompertz' type of density regulation, where $\Delta \ln N$ is a linear function of $\ln N$.

3. For the case of $\theta = 1$ we obtain

$$\begin{aligned}
r(N) &= r_1 \left(1 - \frac{N-1}{K-1}\right) \\
&= r_0 \left(1 - \frac{1}{K}\right) \left(1 - \frac{N-1}{K-1}\right) \\
&= r_0 \left(1 - \frac{1}{K} - \frac{N-1}{K-1} + \frac{1}{K}\frac{N-1}{K-1}\right) \\
&= r_0 \left(1 - \frac{K-1+KN-K-N+1}{K(K-1)}\right) \\
&= r_0 \left(1 - \frac{N(K-1)}{K(K-1)}\right) \\
&= r_0 \left(1 - \frac{N}{K}\right)
\end{aligned}$$

This is the logistic model.

4. The limit as $\theta \to \infty$ (assuming $N < K$) is given by

$$\begin{aligned}
\lim_{\theta \to \infty} r(N) &= \lim_{\theta \to \infty} r_1 \left(1 - \frac{N^\theta - 1}{K^\theta - 1}\right) \\
&= \lim_{\theta \to \infty} r_1 \left(1 - \frac{\frac{1}{K^\theta - 1}}{\frac{1}{N^\theta - 1}}\right) \\
&= r_1
\end{aligned}$$

For $\theta \to \infty$ even a small increase in N above K would produce a growth rate $r(N) \to -\infty$. This is the "roof"-model, where the population grows exponentially until K, but never grow past this limit.

Exercise 3.1

From chapter 1 we know that the process $X_t|X_0$ is approximately normally distributed:

$$\begin{aligned}
X_t|X_0 &\sim \mathrm{N}(X_0 + st, \sigma_s^2 t) \\
s &\approx r - \frac{1}{2}\sigma_e^2 - \frac{1}{2n}\sigma_d^2 \\
\sigma_s^2 &\approx \sigma_e^2 + \frac{\sigma_d^2}{n}
\end{aligned}$$

Hence, the infinitesimal mean and variance are given by

$$\mu_X(x) = \mathrm{E}[\Delta X | X_t] = r - \frac{1}{2}\,\sigma_e^2 - \frac{1}{2e^x}\,\sigma_d^2$$
$$\nu_X(x) = \mathrm{Var}(\Delta X | X_t) = \sigma_e^2 + \frac{\sigma_d^2}{e^x}$$

Exercise 3.2

$$\begin{aligned}
\mu_X(x^*) &= 0 \\
r - \frac{1}{2}\,\sigma_e^2 - \frac{1}{2e^{x^*}}\,\sigma_d^2 &= 0 \\
\frac{1}{2e^{x^*}}\,\sigma_d^2 &= r - \frac{1}{2}\,\sigma_e^2 \\
e^{x^*} &= n^* = \frac{\sigma_d^2}{2r - \sigma_e^2} \\
x^* &= \ln\left(\frac{\sigma_d^2}{2r - \sigma_e^2}\right)
\end{aligned}$$

Exercise 3.3

Here, we use the transformation formulas for diffusion processes, with $X_t = g(N_t) = \ln N_t$, giving

$$\begin{aligned}
\mu_X(x) &= g'(n)\mu(n) + \frac{1}{2}g''(n)\nu(n) \\
&= \frac{1}{n}\,r_1 n\left(1 - \frac{\ln n}{\ln K}\right) - \frac{1}{2n^2}\,\sigma_e^2 n^2 \\
&= r_1\left(1 - \frac{\ln n}{\ln K}\right) - \frac{1}{2}\,\sigma_e^2 \\
&= r_1 - \frac{1}{2}\,\sigma_e^2 - \frac{r_1}{\ln K}\,x \\
\nu_X(x) &= g'(n)^2\nu(n) \\
&= \frac{1}{n^2}\,\sigma_e^2 n^2 \\
&= \sigma_e^2.
\end{aligned}$$

For the OU-process the infinitesimal mean and variance are given by

$$\mu_X(x) = \alpha - \beta\, x$$
$$\nu_X(x) = \sigma_e^2$$

Hence, with $\alpha = r_1 - \sigma_e^2/2$ and $\beta = r_1/\ln K$ we see that X_t is an OU-process.

Exercise 3.4

We use the transformation formula for diffusion processes, with $X_t = g(N_t) = N_t^\theta$, giving

$$\begin{aligned}
\mu_X(x) &= g'(n)\mu(n) + \frac{1}{2}g''(n)\nu(n) \\
&= \theta n^{\theta-1} rn \left(1 - \frac{n^\theta}{K^\theta}\right) + \frac{1}{2}\,\theta(\theta-1)n^{\theta-2}\,\sigma_e^2 n^2 \\
&= \theta r n^\theta - \frac{\theta r n^{2\theta}}{K^\theta} + \frac{1}{2}\,\theta(\theta-1)n^\theta\,\sigma_e^2 \\
&= \theta x \left(r\left[1 - \frac{x}{K^\theta}\right] + \frac{1}{2}\,(\theta-1)\,\sigma_e^2 \right)
\end{aligned}$$

$$\begin{aligned}
\nu_X(x) &= g'(n)^2\nu(n) \\
&= \theta^2 n^{2\theta-2}\,\sigma_e^2 n^2 \\
&= \theta^2 x^2\,\sigma_e^2
\end{aligned}$$

Exercise 3.5

Using the transformation formula for diffusion processes, with $X_t = g(N_t) = N_t^{-\theta}$, we obtain

$$\begin{aligned}
X_t &= g(N_t) = N_t^{-\theta} \\
\mu_X(x) &= g'(n)\mu(n) + \frac{1}{2}g''(n)\nu(n) \\
&= -\theta n^{-\theta-1} rn \left(1 - \frac{n^\theta}{K^\theta}\right) + \frac{1}{2}\,\theta(-\theta-1)n^{-\theta-2}\,\sigma_e^2 n^2 \\
&= -\theta n^{-\theta} r + \frac{r\theta}{K^\theta} - \frac{1}{2}\,\theta(\theta+1)n^{-\theta}\,\sigma_e^2 \\
&= \frac{r\theta}{K^\theta} - \left(r\theta - \frac{1}{2}\theta(\theta+1)\sigma_e^2\right)\,x
\end{aligned}$$

Hence, $\mu_X(x)$ is the infinitesimal mean of an OU-process in the form $\mu_X(x) = \alpha - \beta x$.

Exercise 3.6

In order to obtain a constant infinitesimal variance $\nu_X(x) = 1$ we use the isotrofic scale transformation given by

$$
\begin{aligned}
g(n) &= \int_a^n \frac{1}{\sqrt{\nu_N(z)}}\ \mathrm{d}z \\
&= \int_a^n \frac{1}{\sigma_d \sqrt{z}}\ \mathrm{d}z \\
&= \frac{2}{\sigma_d}\ \left[\sqrt{n} - \sqrt{a}\right]
\end{aligned}
$$

We choose $a = 0$, giving

$$
g(n) = \frac{2\sqrt{n}}{\sigma_d}
$$

Exercise 3.12

The browian motion, which is the process of $X = \ln N$, is defined by

$$
\begin{aligned}
\mu(x) &= s \\
\nu(x) &= \sigma_e^2
\end{aligned}
$$

Choosing $a = 0$ as lower integration limit, $s(x)$ is given by

$$
\begin{aligned}
s(x) &= \exp\left(-2\int_0^x \frac{\mu(z)}{\nu(z)}\,\mathrm{d}\,z\right) \\
&= \exp\left(-2\int_0^x \frac{s}{\sigma_e^2}\,\mathrm{d}\,z\right) \\
&= \exp\left(-\frac{2s}{\sigma_e^2}\,x\right)
\end{aligned}
$$

Using this, $S(x)$ is given by

$$
\begin{aligned}
S(x) &= \int_0^x s(z)\mathrm{d}\,z \\
&= \int_0^x \exp\left(-\frac{2s}{\sigma_e^2}\,z\right)\mathrm{d}\,z \\
&= \begin{cases} \frac{\sigma_e^2}{2s}\left(1 - \exp\left(-\frac{2s}{\sigma_e^2}\,x\right)\right) & s \neq 0 \\ x & s = 0 \end{cases}
\end{aligned}
$$

In order to find $u(x_0)$, we evaluate $S(a)$, $S(b)$ and $S(x_0)$ for $a = 0$ and $b \to \infty$, and consider the different cases $s > 0$, $s < 0$, and $s = 0$.

$$
\begin{aligned}
S(0) &= \; 0 \quad \forall s \\
\lim_{b\to\infty} S(b) &= \begin{cases} -\infty & s < 0 \\ \frac{\sigma_e^2}{2s} & s > 0 \\ \infty & s = 0 \end{cases} \\
S(x_0) &= \begin{cases} \frac{\sigma_e^2}{2s}\left(1 - \exp\left(-\frac{2s}{\sigma_e^2}\, x_0\right)\right) & s \neq 0 \\ x & s = 0 \end{cases}
\end{aligned}
$$

This gives

$$
\begin{aligned}
u(x_0) &= \frac{S(x_0)}{S(b)} \\
&= \begin{cases} 0 & s \leq 0 \\ 1 - \exp\left(-\frac{2s}{\sigma_e^2}\, x_0\right) & s > 0 \end{cases}
\end{aligned}
$$

Exercise 3.15

The expression $G(x, x_0)\Delta x$ represents the expected time the population process will spend in a small interval $(x, x + \Delta x)$, before it is absorbed at a or b (see figure 2).

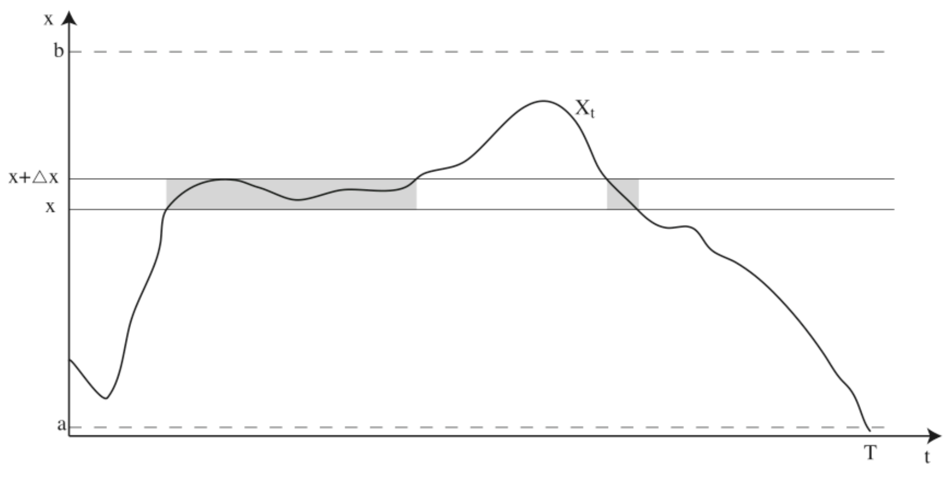

Figure 2: Illustration of how much time the process X_t spends in an interval $(x, x + \Delta x)$.

Let $T_x = G(x, x_0)\Delta x$. In the interval $(x, x+\Delta x)$ the value of the function $h(X_t)$ will be approximately a constant $h(x)$ as Δx becomes small. Then $\mathrm{E}\left[\int_0^T h(X_t)\,\mathrm{d}t\right]$ will become equal to $T_x h(x)$ in the interval $(x, x+\Delta x)$. In order to find the expectation of all x we summarize over all possible intervals between a and b. Letting $\Delta x \to 0$ we obtain the integral.

Exercise 3.19

Firstly, we have

$$Pr(N_t > n) = Pr(\ln N_t > \ln n) = Pr(X_t > x)$$

hence we may study $X_t = \ln N_t$ instead of N. Let X_t and X_t^* the processes without and with a barrier, respectively. Then X_t is normally distributed with expectation $x_0 + st$ and variance $\sigma_e^2 t$, and

$$Pr(X > x) = 1 - \Phi\left(\frac{x - x_0 - st}{\sigma_e\sqrt{t}}\right)$$

For the process with barrier we have (from exercise 3.18)

$$Pr(X_t^* > x) = 1 - \Phi\left(\frac{x - x_0 - st}{\sigma_e\sqrt{t}}\right) - \exp\left(-\frac{2x_0 s}{\sigma_e^2}\right)\Phi\left(\frac{st - x - x_0}{\sigma_e\sqrt{t}}\right)$$

Hence, the probability is lower if the extinction barrier is $N = 1$ $(X = 0)$. The difference between the two probabilities is given by

$$Pr(X > x) - Pr(X_t^* > x) = \exp\left(-\frac{2x_0 s}{\sigma_e^2}\right)\Phi\left(\frac{st - x - x_0}{\sigma_e\sqrt{t}}\right)$$